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The Series Product for Gaussian Quantum Input Processes

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Abstract

We present a theory for connecting quantum Markov components into a network with quantum input processes in a Gaussian state (including thermal and squeezed). One would expect on physical grounds that the connection rules should be independent of the state of the input to the network. To compute statistical properties, we use a version of Wicks' Theorem involving fictitious vacuum fields (Fock space based representation of the fields) and while this aids computation, and gives a rigorous formulation, the various representations need not be unitarily equivalent. In particular, a naive application of the connection rules would lead to the wrong answer. We establish the correct interconnection rules, and show that while the quantum stochastic differential equations of motion display explicitly the covariances (thermal and squeezing parameters) of the Gaussian input fields We introduce the Wick-Stratonovich form which leads to a way of writing these equations that does not depend on these covariances and so corresponds to the universal equations written in terms of formal quantum input processes. We show that a wholly consistent theory of quantum open systems in series can be developed in this way, and as required physically, is universal and in particular representation-free.

Keywords: Gaussian Wick Theorem, Wick-Stratonovich Form, Quantum Gaussian Feedback Networks.

1 Introduction

The quantum input-output theory has had an immense impact on quantum optics, and in recent years has extended to opto-mechanical systems and beyond. The prospect of routing the inputs through a network, or indeed using feedback has lead to a burgeoning field of quantum feedback control [1]-[5]. The development of a systems engineering approach to quantum technology has benefited from having a systematic framework in which traditional open quantum systems models can be combined according to physical connection architectures.

The initial work on how to cascade two quantum input-output systems can be traced back to Gardiner [6] and Carmichael [7]. More generally, the authors have introduced the theory of *Quantum Feedback Networks* (QFN) which generalizes this to include cascading, feedback, beam-splitting and general scattering of inputs, etc., [8], [9]. One of the basic constructs is the series product which gives the instantaneous feedforward limit of two components connected in series via quantum input processes: in fact, the systems need not necessarily be distinct and the series product generalizes cascading by allowing for feedback. The original work was done for input processes where the input fields where in the Fock vacuum field state. A generalization to squeezed fields and squeezing components has been given [10], however this was restricted to the case of linear coupling and dynamics: there it was shown that the resulting transform analysis could be applied in a completely consistent manner. More recent work has shown that non-classical states for the input fields, such as shaped single-photon or multi-photon states, or cat states of coherent fields, may in principle be generated from signal models [11], [12] - that is, where a field in the Fock vacuum state was passed through an ancillary dynamical system (the signal generator) which is then to be cascaded to the desired system. Quantum feedback network (QFN) theory concerns the interconnection of open quantum systems. The interconnections are mediated by quantum fields in the input-output theory, [13, 8, 9]. The idea is that an output from one node is fed back in as input to another (not necessarily distinct) node, the simplest case being the cascade connection (e.g., light exiting one cavity being directed into another). The components are specified by Markovian models determined by SLH parameters which describe the self-energy of the system and how the system interacts with the fields (via idealized Jaynes-Cummings type interactions and scattering).

Here we turn to the problem of the general class of Gaussian states for quantum fields. This includes thermal fields, and of course squeezed fields. In principle, these may be approximated as the output of a degenerate parametric amplifier (DPA) driven by vacuum input, see [13]. In a sense, we have that a singular DPA may serve is the appropriate signal generator to modify a vacuum field into a squeezed field before passing into a given network. We will exploit this in the paper, however, we will have to pay attention to the operator ordering problem when inserting these approximations into quantum dynamical equations of motion and input-output relations.

The programme turns out to be rather more involved than one might expect at first glance. It is always possible to represent a collection of d Gaussian fields using $2d$ vacuum fields (a Bogoliubov transformation!) and one might hope that the corresponding connection rules applied to the representation in terms of vacuum fields would agree with the intuitive rules one would desire. This turns out not to be the case, and the various feedback constraints cannot be naively applied to the representing fields: the reason is that the representations are a linear combination of creation and annihilation operators for the representing vacuum fields, and we have broken the Wick ordered form of the original equations.

If applied naively, the series product would predict a contribution to the global network model that depended on the covariance parameters of the state. From the physical point of view, this ought to be spurious. In comparison with classical analog linear electronics, we see that the components (e.g. resistors, capacitors, inductors) are described by impedances. When components are interconnected to form a network, the network may be described by an equivalent impedance, derived through an application of Kirchhoff laws. Impedances do not depend on the applied currents or voltages, and are therefore intrinsic to the device or network. Similarly the rules for connecting a quantum feedback network should be intrinsic, and not depend on the state of the noise fields.

2 Background and Problem statement

Let us begin in the concrete setting of the quantum stochastic calculus of Hudson and Parthasarathy [15] with a fixed initial space $\mathfrak{h}_0$ and a noise space that is the (Bose) Fock space over $\mathbb{C}^d$-valued L^2-functions on the time interval $[0,\infty)$. In the language of Hudson and Parthasarathy, we have a multiplicity space of dimension d and we select an orthonormal basis which determines d channels. We denote by $A_k(t)$, $A_k(t)^*$, and $\Lambda_{jk}(t)$ the processes of annihilation, creation (for channel j) and scattering (from channel k to channel j). In the following, we shall introduce an Einstein summation convention for repeated channel indices. We will deal with the class of quantum stochastic integrals processes satisfying the appropriate conditions of local integrability, square-integrability [15] without explicit reference. We have for instance the QSDE

$$dX(t) = x_{jk}(t)\,d\Lambda_{jk}(t) + x_{j0}(t)\,dA_j(t)^* + x_{0k}dA_k(t) + x_{00}(t)\,dt \tag{1}$$

where the coefficients are adapted and the increments are (quantum) Itō. We have the quantum Itō product formula

$$d(X(t)\,Y(t)) = dX(t)\;Y(t) + X(t)\;dY(t) + dX(t)\;dY(t) \tag{2}$$

where the Itō correction comes from the quantum Itō table [15]

$$\begin{aligned} d\Lambda_{jk}(t)\;d\Lambda_{lm}(t) &= \delta_{kl}d\Lambda_{jm}(t)\,, \quad d\Lambda_{jk}(t)\;dA_l(t)^* = \delta_{kl}dA_j(t)^*\,, \\ dA_k(t)\;d\Lambda_{lm}(t) &= \delta_{kl}dA_m(t)\,, \quad dA_j(t)\;dA_k = \delta_{jk}dt, \end{aligned} \tag{3}$$

with all other products of the fundamental increments vanishing.

Definition 1 *Definition The Stratonovich integral is defined algebraically via*

$$X(t)\circ dY(t) = X(t)\,dY(t) + \frac{1}{2}dX(t)\;dY(t) \tag{4}$$

$$dX(t)\circ Y(t) = dX(t)\;Y(t) + \frac{1}{2}dX(t)\;dY(t)\,. \tag{5}$$

This turns out to be equivalent to a mid-point rule [17].

If we consider the QSDE $dU\left(t\right)=-idE\left(t\right)\circ U\left(t\right)$, with $U\left(0\right)$ the identity and $E\left(t\right)=E_{jk}\Lambda_{jk}\left(t\right)+E_{j0}B_{j}\left(t\right)^{\ast}+E_{0k}B_{k}\left(t\right)+E_{00}$ a self-adjoint quantum stochastic integral process, then we may convert to the Itō form to get

$$dU\left(t\right)=\left\{ \left(S_{jk}-\delta_{jk}\right)d\Lambda_{jk}\left(t\right)+L_{j}dA_{j}^{\ast}\left(t\right)-L_{j}^{\ast}S_{jk}dA_{k}\left(t\right)+Kdt\right\} U\left(t\right), \quad (6)$$

where (setting $E_{\ell\ell}$ to be the $d\times d$ matrix with entries $E_{jk)}$

$$S=\left[\begin{array}{ccc} S_{11} & \cdots & S_{1d}\\ \vdots & \ddots & \vdots\\ S_{d1} & \cdots & S_{dd}\end{array}\right]=\frac{I-\frac{i}{2}E_{\ell\ell}}{I+\frac{i}{2}E_{\ell\ell}} \quad (7)$$

is called the matrix of scattering coefficients unitary (that is, $S_{jk}^{\ast}S_{jl}=\delta_{kl}=S_{lj}S_{kj}^{\ast}$),

$$L=\left[\begin{array}{c} L_{1}\\ \vdots\\ L_{d}\end{array}\right]=\frac{i}{I+\frac{i}{2}E_{\ell\ell}}\left[\begin{array}{c} E_{10}\\ \vdots\\ E_{d0}\end{array}\right] \quad (8)$$

which is the column vector of coupling operators, and

$$K=-\frac{1}{2}L_{k}^{\ast}L_{k}-iH, \quad (9)$$

where H is the Hamiltonian ($H^{\ast}=H=E_{00}+\frac{1}{2}E_{0j}\left[\mathrm{Im}\left\{ \frac{1}{I+\frac{i}{2}E_{\ell\ell}}\right\} \right]_{jk}E_{k0}$). For simplicity we will assume that the terms S_{jk},L_{j} and H are bounded operators on the system Hilbert space $\mathfrak{h}_{0}$.

We generally refer to the triple $\mathbf{G}\sim\left(S,L,H\right)$ as the Hudson-Parthasarathy parameters, or informally the "SLH" parameters specifying the model. The unitary process they generate may be denoted as $U^{\mathbf{G}}\left(t\right)$ if we wish to emphasize the dependence on these parameters.

For X an operator of the initial space, we introduce $j_{t}\left(X\right)=U\left(t\right)^{\ast}X\,U\left(t\right)$ and from the quantum Itō rule obtain the Heisenberg-Langevin equation

$$dj_{t}(X)=j_{t}\left(\mathcal{L}_{jk}X\right)\,d\Lambda_{jk}+j_{t}(\mathcal{L}_{j0}X)\,dA_{j}^{\ast}+j_{t}(\mathcal{L}_{0k}X)\,dA_{k}+j_{t}(\mathcal{L}_{00}X)dt \quad (10)$$

where

$$\mathcal{L}_{jk}X=S_{lj}^{\ast}XS_{lk}-\delta_{jk}X,\quad\mathcal{L}_{j0}X=S_{lj}^{\ast}\left[X,L_{l}\right],\quad\mathcal{L}_{0k}X=\left[L_{l}^{\ast},X\right]S_{lk} \quad (11)$$

and the Lindblad generator $\mathcal{L}_{00}\equiv\mathcal{L}$ is

$$\mathcal{L}X=\frac{1}{2}L_{k}^{\ast}\left[X,L_{k}\right]+\frac{1}{2}\left[L_{k}^{\ast},X\right]L_{k}-i\left[X,H\right]. \quad (12)$$

The maps $\mathcal{L}_{\alpha\beta}$ are known as the *Evans-Hudson super-operators*. We shall occasionally write $j_{t}^{\mathbf{G}}\left(X\right)$ for the dynamical flow of X when we wish to emphasis the dependence on the SLH parameters $\mathbf{G}$.

Let us now write the input processes as $A_{\mathrm{in},j}(t) = A_j(t)$ and introduce the output processes as $A_{\mathrm{out},j}(t) = U(t)^* A_{\mathrm{in},j}(t) U(t)$ then from the quantum Itō rule we see that

$$dA_{\mathrm{out},j}(t) = j_t(S_{jk})\, dA_{\mathrm{in},k}(t) + j_t(L_l)\, dt. \tag{13}$$

2.1 Thermal Fields

Considering the single channel ($d = 1$) case for the moment, we may introduce non-Fock quantum stochastic processes as follows [18]. For $n > 0$, we set

$$B(t) = \sqrt{n+1}A_+(t) + \sqrt{n}A_-(t)^*, \quad \tilde{B}(t) = \sqrt{n}A_+(t) + \sqrt{n+1}A_-(t)^* \tag{14}$$

which are canonical fields on the Fock space with a pair of channels labeled as $k = \pm$. In fact, the map $(A_+, A_-) \mapsto \left(B, \tilde{B}\right)$ is a Bogoliubov transformation with inverse

$$\left[\begin{array}{c} A_+ \\ A_- \end{array}\right] = \left[\begin{array}{cc} \sqrt{(n+1)} & -\sqrt{n} \\ -\sqrt{n} & \sqrt{(n+1)} \end{array}\right] \left[\begin{array}{c} B \\ \tilde{B} \end{array}\right]. \tag{15}$$

This is of course based on an Araki-Woods representation of the fiels [19]. As is well known, these transformation cannot be implemented unitarily. However, from a quantum optics point of view, devices transforming or even squeezing fields in this manner are frequently considered, and it is useful to imagine a hypothetical device - a Bogoliubov box - performing such a canonical transformation on our idealized fields.

Ignoring the second process $\tilde{B}$, we obtain the non-Fock quantum Itō table

$$dB(t)\, dB(t)^* = (n+1)\, dt, \quad dB(t)^*\, dB(t) = ndt. \tag{16}$$

It problematic (read impossible) to incorporate a scattering process Λ into this table. We refer to B as non-Fock quantum noise.

We need to drop the scattering term from the unitary evolution equation, i.e. set $S \equiv I$, and with

$$L = \left[\begin{array}{c} L_+ \\ L_- \end{array}\right] = \left[\begin{array}{c} \sqrt{n+1}L \\ -\sqrt{n}L^* \end{array}\right] \tag{17}$$

we have

$$\begin{aligned} dU(t) &= \left\{ L dB(t)^* - L^* dB(t) + K^{\mathrm{th}} dt \right\} U(t) \\ &= \left\{ L_j dA_j^*(t) - L_j^* dA_j(t) + K dt \right\} U(t), \end{aligned} \tag{18}$$

where

$$K^{\mathrm{th}} = -\frac{1}{2}L_+^* L_+ - \frac{1}{2}L_-^* L_- - iH = -\frac{n+1}{2}L^* L - \frac{n}{2}LL^* - iH. \tag{19}$$

For the flow, we need that the Hudson-Evans super-operator associated with the scattering terms are trivial. This is the case when the entries of the scattering matrix S are (e.g. scalars) commuting with operators of the initial space, but we can get away without assuming that $\left[\begin{array}{cc} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{array}\right]$ is the identity. By inspection we find that flow equation will take the form

$$dj_t(X) = j_t([X,L])\, S^* dB(t)^* + j_t([L,X])\, S dB(t) + j_t\left(\mathcal{L}^{\mathrm{th}} X\right) dt \tag{20}$$

if and only if we take $\left[\begin{array}{cc} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{array}\right] \equiv \left[\begin{array}{cc} S & 0 \\ 0 & S^* \end{array}\right]$ - otherwise we obtain the other noise $\tilde{B}$ - and in which case the Lindbladian is

$$\begin{aligned} \mathcal{L}^{\mathrm{th}} X &= \frac{1}{2}\left[L_+^*, X\right] L_+ + \frac{1}{2} L_+^* \left[X, L_+\right] + \frac{1}{2}\left[L_-^*, X\right] L_- + \frac{1}{2} L_-^* \left[X, L_-\right] - i[X,H] \\ &= \frac{n+1}{2}\left\{\left[L^*, X\right] L + L^* \left[X, L\right]\right\} + \frac{n}{2}\left\{\left[L^*, X\right] L + L^* \left[X, L\right]\right\} - i[X,H] \end{aligned} \tag{21}$$

2.2 The Series Product - Vacuum Inputs

In [9] the authors introduce a rule for combining SLH models in series. For instance, we have the output of the $\mathbf{G}_{\mathscr{A}} \sim (S_{\mathscr{A}}, L_{\mathscr{A}}, H_{\mathscr{A}})$ fed instantaneously as input to $\mathbf{G}_{\mathscr{B}} \sim (S_{\mathscr{B}}, L_{\mathscr{B}}, H_{\mathscr{B}})$ and it is shown that this is equivalent to the model generated by

$$\begin{aligned} \mathbf{G}_{\mathscr{B}} \triangleleft \mathbf{G}_{\mathscr{A}} &\sim (S_{\mathscr{A}}, L_{\mathscr{A}}, H_{\mathscr{A}}) \triangleleft (S_{\mathscr{B}}, L_{\mathscr{B}}, H_{\mathscr{B}}) \\ &= \Big(S_{\mathscr{B}} S_{\mathscr{A}}, L_{\mathscr{B}} + S_{\mathscr{B}} L_{\mathscr{A}}, H_{\mathscr{A}} + H_{\mathscr{B}} + \mathrm{Im}\left\{ L_{\mathscr{B}}^* S_{\mathscr{B}} L_{\mathscr{A}} \right\} \Big). \end{aligned} \tag{22}$$

Here $\mathrm{Im}\{C\}$ means $\frac{1}{2i}(C - C^*)$.We note that every model may be written as a purely scattering component and a non-scattering component in series, since we have the law $(S,L,H) = (I,L,H) \triangleleft (S,0,0)$.

We should remark that it is not necessary to view the two systems $\mathscr{A}$ and $\mathscr{B}$ as separate systems - specifically, in the derivation of the series product[9] it is not assumed that the $\mathscr{A}$ and $\mathscr{B}$ operators need commute!

2.3 Statement of the Problem

If we wish to have a pair of systems $\mathscr{A}$ and $\mathscr{B}$ (both accepting d inputs) in series, then we obtain an equivalent Markov model in the limit where the intervening connection is instantaneous. Let $L_{\mathscr{A}}$ be the column of the d operators $L_{\mathscr{A},k}$, $k = 1, \cdots, d$, and similar for system $\mathscr{B}$. The series product says that the equivalent model has coupling $L_{\mathscr{A}} + L_{\mathscr{B}}$ and Hamiltonian

$$H_{\mathscr{A}} + H_{\mathscr{B}} + \mathrm{Im}\left\{ L_{\mathscr{B}}^* L_{\mathscr{A}} \right\}. \tag{23}$$

Suppose we were to apply the series product to two systems with the same single thermal input B, and try and describe this as a series connection using the two vacuum inputs A_+ and A_-. Naively applying the series product to the construction in the $A_\pm$ format leads to the correct rule $L_{\mathscr{A}} + L_{\mathscr{B}}$ for the coupling terms, but

$$H_{\mathscr{A}} + H_{\mathscr{B}} + \mathrm{Im}\,\{L_{\mathscr{B}}^* L_{\mathscr{A}}\} + n\mathrm{Im}\,[L_{\mathscr{B}}^*, L_{\mathscr{A}}]\,. \tag{24}$$

We have picked up an n-dependent term. For pure cascading, the systems $\mathscr{A}$ and $\mathscr{B}$ are distinct and so $[L_{\mathscr{B}}^*, L_{\mathscr{A}}] = 0$. However, the series product should also apply to the situation where the systems share degrees of freedom. In such cases the additional term is physically unreasonable as it depends on the state of the noise.

It is not immediately obvious what is wrong with the construction. Going to the double Fock vacuum representations and then using the vacuum version of the series product would seem a reasonable thing to do. However, a fully quantum description would involve the $\tilde{B}$ fields as well, and at a schematic level this would involve one or more Bogoliubov boxes - something conspicuously. We will give the correct procedure in this paper.

3 Multi-Dimensional Gaussian Processes

3.1 Notation

We will use the symbol $\triangleq$ to signify a defining equation. We will denote the operations of complex conjugation, hermitean conjugation, and more generally adjoint by *. For $X = [x_{ij}]$ an $n \times m$ array with complex-valued entries, or more generally operator-valued entries, we write X^* for the $m \times n$ array obtained by transposition of the array and conjugation of the entries: that is the ij entry is x_{ji}^*. The transpose alone will be denotes as $X^\top$, that is the $m \times n$ array with ij entry x_{ji}. We will also use the notation $X^\# = (X^\top)^*$ which is the $n \times m$ array with ij entry x_{ij}^*.

3.2 Finite Dimensional Gaussian States

Let $a_1, \cdots, a_d$ be the annihilation operators for d independent oscillators. We consider a mean zero Gaussian state with second moments

$$n_{ij} = \langle a_i^* a_j \rangle, \quad m_{ij} = \langle a_i a_j \rangle, \tag{25}$$

which we assemble into a hermitean $d \times d$ matrix, N, with entries $n_{ji}^* = n_{ij}$, and a symmetric matrix M is the $d \times d$ matrix with entries $m_{ij} = m_{ji}$. The *covariance matrix* is

$$F = \left[\begin{array}{cc} I + N^\top & M \\ M^* & N \end{array} \right]. \tag{26}$$

In order to yield mathematically correct variances, we must have both F and N positive. The vacuum state is characterized by having $N = M = 0$, that is

$$F_{\mathrm{vac}} \equiv \left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right]. \tag{27}$$

The covariance matrix F defined by (27) must be positive semi-definite, as will be the matrices N and $I+N^{\top}$. We must also have $\mathrm{ran}(M) \subseteq \mathrm{ran}\big(I+N^{\top}\big)$ and $MN^{-}M^{*} \leq I+N$, where N^{-} is the Moore-Penrose inverse of N.

A linear transformation of the form

$$\tilde{a} = Ua + Va^{\#}, \tag{28}$$

that is $\tilde{a}_j = \sum_k (U_{jk} a_k + V_{jk} a_k^{*})$, is called a *Bogoliubov transformation* if we have again the canonical commutation relations for the primed operators.

The transformation $\tilde{a} = Ua + Va^{\#}$ is Bogoliubov if and only if the following identities hold $UU^{*} = I + VV^{*}, \quad UV^{\top} = VU^{\top}$.

This is easily established by inspection, as are the following.

Lemma 2 *Lemma Let $\tilde{a} = Ua + Va^{\#}$ be a Bogoliubov transformation, then the covariance matrix for $\tilde{a}$ is*

$$\tilde{F} = WFW^{\dagger} \tag{29}$$

where $W = \Delta(U,V)$. In particular, the new matrices are

$$\begin{aligned} N' &= V^{\#}V^{\top} + V^{\#}N^{\top}V^{\top} + U^{\#}M^{*}V^{\top} + V^{\#}MU^{\top} + U^{\#}NU^{\top}, \\ M' &= UV^{\top} + UN^{\top}V^{\top} + VM^{*}V^{\top} + UM^{*}U^{\top} + VNU^{\top}. \end{aligned} \tag{30}$$

Lemma 3 *Lemma Given a_{vac} with the choice of the vacuum state, the Bogoliubov transformation $a = Ua_{\mathrm{vac}} + Va_{\mathrm{vac}}^{\#}$ leads to operators with the covariance matrix*

$$F = WF_{\mathrm{vac}}W^{*} = \begin{bmatrix} I+N^{\top} & M \\ M^{*} & N \end{bmatrix} \tag{31}$$

where $W = \Delta(U,V)$ and

$$N = V^{\#}V^{\top}, \quad M = UV^{\top}. \tag{32}$$

We note that the determinant of the covariance matrix is preserved under Bogoliubov transformations. In particular, if we have $F = WF_{\mathrm{vac}}W^{*}$, as in the last Proposition, then F must also be singular. This means that if we wish to obtain a given covariance matrix F for d modes by a Bogoliubov transformation of vacuum modes, we will typically need a larger number D of these modes with F being a sub-block of a transformed matrix $WF_{\mathrm{vac}}W^{*}$. The example in the Theorem shows that in order to obtain the $d=1$ covariance

$$F = \begin{bmatrix} 1+n & 0 \\ 0 & n \end{bmatrix} \tag{33}$$

we need a Bogoliubov transformation of $D=2$ modes. We remark that we may obtain the covariance

$$F = \begin{bmatrix} 1+n & m \\ m^{*} & n \end{bmatrix}, \tag{34}$$

with the constraint $|m|^2 \leq n(n+1)$ ensuring positivity, from 2 vacuum modes via [21, 20]

$$\tilde{a} = \sqrt{n+1-\frac{1}{n}|m|^2}a_1 + \sqrt{n}a_2^* + \frac{m}{\sqrt{n}}a_2. \tag{35}$$

The maximal case $|m|^2 = n(n+1)$ may be obtained from a *single* mode a_1 via $a = \sqrt{n+1}a_1 + e^{i\theta}\sqrt{n}a_1^*$ where $m \equiv \sqrt{n(n+1)}e^{i\theta}$.

3.3 Quantum Ito Calculus: Gaussian Noise

One would like to extend this to non-vacuum inputs, in particular, those with general flat power Gaussian states for the noise. (We restrict to a single noise channel for transparency but the generalization is straightforward enough.) It is possible to construct noises having the following quantum Itō table

$$\begin{aligned} dB_i dB_j^* &= (n_{ji} + \delta_{ij})\,dt, \quad dB_i^* dB_j = n_{ij}dt, \\ dB_i dB_j &= m_{ij}dt, \quad dB_i^* dB_j^* = m_{ji}^* dt, \end{aligned} \tag{36}$$

where $N = [n_{ij}]$ and $M = [m_{ij}]$ have the same properties and constraints as introduced above.

In reality, we are assuming that the fields $B_j(t)$ correspond to a representation on a double Fock space, say,

$$B(t) = U \begin{bmatrix} A_+(t) \otimes I \\ I \otimes A_-(t) \end{bmatrix} + V \begin{bmatrix} A_+(t)^{\#} \otimes I \\ I \otimes A_-(t)^{\#} \end{bmatrix} \tag{37}$$

where $A_k(t) = \begin{bmatrix} A_{k,1}(t) \\ \vdots \\ A_{k.d}(t) \end{bmatrix}$ are copies of the Fock fields encountered above, and where $N = V^{\#}V, M = UV^{\top}$ as in Proposition 3.

The underlying mathematical problem is that we are trying to implement a canonical transformation that is not inner [22, 23, 24]- specifically the various representations for different pairs (N, M) are not unitarily equivalent.

Instead we must restrict to QSDE models in the general Gaussian case which are driven by B and B^* only. We in fact find the class of QSDEs

$$dU(t) = \left\{ L_k dB_k^*(t) - L_k^* dB_k(t) + K^{(N,M)}dt \right\} U(t) \tag{38}$$

generating unitaries and we now require that

$$K^{(N,M)} = -\frac{1}{2}(\delta_{ij} + n_{ji})L_i^* L_j - \frac{1}{2}n_{ij}L_i L_j^* + \frac{1}{2}m_{ij}L_i^* L_j^* + \frac{1}{2}m_{ji}^* L_i L_j - iH, \tag{39}$$

with H again self-adjoint.

Let us denote the conditional expectation from the algebra of operators on the system-tensor-Fock Hilbert space down to the system operators (i.e., the partial trace over the Gaussian state) as $\mathbb{E}_{(N,M)}[\cdot|\text{sys}]$. As the differentials $dB_k(t)$

and $dB_k(t)^*$ are Itō (future pointing) their products with adapted operators will have conditional expectation zero. Therefore

$$\mathbb{E}_{(N,M)}\left[dU_t|\text{sys}\right] = K^{(N,M)}\,\mathbb{E}_{(N,M)}\left[U_t|\text{sys}\right]\,dt \tag{40}$$

and we deduce that

$$\mathbb{E}_{(N,M)}\left[U_t|\text{sys}\right] = e^{tK^{(N,M)}}. \tag{41}$$

The corresponding Heisenberg-Langevin equations are of the form

$$dj_t(X) = j_t([X,L_k])dB_k^* + j_t([L_k^*,X])dB_k + j_t(\mathcal{L}^{(N,M)}X)dt \tag{42}$$

where the new Lindbladian is

$$\begin{aligned}\mathcal{L}^{(N,M)}X &= \frac{1}{2}(\delta_{ij}+n_{ji})\big\{L_i^*\left[X,L_j\right]+\left[L_i^*,X\right]L_j\big\}\\ &\quad +\frac{1}{2}n_{ij}\big\{L_i\left[X,L_j^*\right]+\left[L_i,X\right]L_j^*\big\}\\ &\quad -\frac{1}{2}m_{ij}\big\{L_i^*\left[X,L_j^*\right]+\left[L_i^*,X\right]L_j^*\big\}\\ &\quad -\frac{1}{2}m_{ji}^*\big\{L_i\left[X,L_j\right]+\left[L_i,X\right]L_j\big\}-i\left[X,H\right].\end{aligned} \tag{43}$$

Likewise, we find that

$$\mathbb{E}_{(N,M)}\left[j_t(X)\,|\text{sys}\right] = e^{t\mathcal{L}^{(N,M)}}X. \tag{44}$$

A little algebra allows us to relate these to the vacuum expressions:

$$K^{(N,M)} = K - \frac{1}{2}n_{ji}L_i^*L_j - \frac{1}{2}n_{ij}L_iL_j^* + \frac{1}{2}m_{ij}L_i^*L_j^* + \frac{1}{2}m_{ji}^*L_iL_j, \tag{45}$$

$$\begin{aligned}\mathcal{L}^{(N,M)}X &= \mathcal{L}X + \frac{1}{2}n_{ji}\big\{L_i^*\left[X,L_j\right]+\left[L_i^*,X\right]L_j\big\}\\ &\quad +\frac{1}{2}n_{ij}\big\{L_i\left[X,L_j^*\right]+\left[L_i,X\right]L_j^*\big\}\\ &\quad -\frac{1}{2}m_{ij}\big\{L_i^*\left[X,L_j^*\right]+\left[L_i^*,X\right]L_j^*\big\}\\ &\quad -\frac{1}{2}m_{ji}^*\big\{L_i\left[X,L_j\right]+\left[L_i,X\right]L_j\big\}\\ &\equiv \mathcal{L}X + \frac{1}{2}n_{ji}\big\{\left[L_i^*,\left[X,L_j\right]\right]+\left[\left[L_i^*,X\right],L_j\right]\big\}\\ &\quad +\frac{1}{2}m_{ij}\left[L_j^*\left[L_i^*,X\right]\right]+\frac{1}{2}m_{ij}^*\left[\left[X,L_i\right],L_j\right].\end{aligned} \tag{46}$$

4 Representation-Free Form

Returning to the problem stated in the Introduction, we have that *all* the $U_t^{(N,M)}$ arise from the *same* physical dynamical evolution U_t, and the dynamics show not depend on the state! The $U_t^{(N,M)}$ unfortunately belong to representations that are *not* generally unitarily equivalent! There should be some sense in which the QSDEs for the various $U_t^{(N,M)}$ should in some sense be equivalent. These QSDEs will depend explicitly on the state parameters (N,M) of the input field, but what we would like to do is to show that there is nevertheless a representation-free version of each of these QSDEs in each fixed representation.

We now show that there is a way of presenting the unitary (38) and Heisenberg (43) QSDEs so as to be independent of the state parameters (N,M).

Theorem 4 *Theorem* ***(Representation-Free Form)*** *The non-Fock QSDEs (38) and (43) may be written in the equivalent Stratonovich forms*

$$\begin{aligned} dU &= dA_k^* \circ L_k U - L_k^* U \circ dA_k + KU\left(t\right) \circ dt, &(47)\\ dj_t(X) &= dA_k^* \circ j_t([X,L_k]) + j_t([L_k^*,X]) \circ dA_k + j_t(\mathcal{L}X) \circ dt, &(48)\end{aligned}$$

respectively, where K and $\mathcal{L}$ are the Fock representation expressions (9) and (12).

Proof. We first observe that

$$dB_k^* \circ L_k U = dB_k^* L_k U + \frac{1}{2} dB_k^* L_k dU \tag{49}$$

and substituting the QSDE (38) for dU and using the quantum Itō table (36) gives

$$dB_k^* \circ L_k U = L_k U dB_k^* + \frac{1}{2} L_k \left(m_{kj}^* L_j - n_{kj} L_j^* \right) U dt, \tag{50}$$

and similarly

$$-L_k^* U \circ dB_k = -L_k^* U dB_k - \frac{1}{2} L_k^* dU dB_k = -L_k^* U dB_k - \frac{1}{2} L_k^* \left(n_{jk} L_j - m_{ki} L_j^* \right) dt. \tag{51}$$

Combining these terms and using the identity (45) shows that (47) is equivalent to (38).

For the Heisenberg equation, we first note that

$$\begin{aligned} dB_k^* \circ j_t([X,L_k]) &= dB_k^* j_t([X,L_k]) + \frac{1}{2} dB_k^* dj_t([X,L_k]) \\ &= j_t([X,L_k]) dB_k^* \\ &\quad + \frac{1}{2} dB_k^* \left\{ j_t([[X,L_k],L_j]) dB_j^* + j_t\left(\left[L_j^*, [X,L_k] \right] \right) dB_j \right\} \\ &= j_t([X,L_k]) dB_k^* + j_t \Big(\frac{1}{2} m_{kj}^* \left[[X,L_k], L_j \right] + \frac{1}{2} n_{kj} \left[L_j^*, [X,L_k] \right] \Big) dt, \end{aligned} \tag{52}$$

and similarly

$$j_t([L_k^*,X])\circ dB_k = j_t([L_k^*,X])dB_k + j_t\left(\frac{1}{2}n_{jk}\left[\,[L_k^*,X]\,,L_j\right] + \frac{1}{2}m_{jk}\left[L_j^*,\left[L_k^*,X\right]\,\right]\right)dt. \tag{53}$$

Combining these terms and using the identity (46) shows that (48) is equivalent to (43). ■

Note that in both equations (47) and (48) the Stratonovich differentials occur in Wick order relative to the integrand terms. What is remarkable about these relations is that they are structurally the same as the Fock vacuum form of the QSDEs with $S=I$. We say that the equations (47) and (48) are *representation-free* in the sense that they do not depend on the parameters N and M determining the state of the noise.

5 White Noise Description

We now present a more formal, but insightful account of quantum stochastic processes. Consider a collection of quantum noise input processes $\{b_k(t) : t \in \mathbb{R}, k = 1, \cdots, d\}$ obeying the commutation relations

$$[b_j(t), b_k^*(s)] = \delta(t-s), \qquad \left[b_j^*(t), b_k^*(s)\right] = [b_j(t), b_k(s)] = 0. \tag{54}$$

We wish to model the interaction of a quantum mechanical system driven by these processes, and to this end introduce a unitary dynamics given by

$$U(t) = \vec{\mathbf{T}}\exp\left\{-i\int_0^t \Upsilon_s ds\right\} \tag{55}$$

where (with an implied summation convention with range $1,\cdots,d$)

$$-i\Upsilon_t = L_k\otimes b_k^*(t) - L_k^*\otimes b_k(t) - iH\otimes I. \tag{56}$$

Here L_k and $H=H^*$ are system operators. The time ordering $\vec{\mathbf{T}}$ is understood in the usual sense of a Dyson series expansion. From this we may arrive at

$$\dot{U}(t) = L_k b_k^*(t)\,U(t) - L_k^* b_k(t)\,U(t) - iHU(t). \tag{57}$$

We claim that $U(t)$ should correspond to the evolution operator for $\mathbf{G}\sim(S=I,L,H)$ without due reference to a particular state for the noise. If we fix the state, say the vacuum, then we use Wick ordering to compute the partial expectations with respect to that state.

To see how to proceed, let us consider a general quantum stochastic integral $X(t)$ described by a formal equation

$$\dot{X}(t) = b_j(t)^*\,x_{jk}(t)\,b_k(t) + b_j(t)^*\,x_{j0}(t) + x_{0k}(t)\,b_k(t) + x_{00}(t). \tag{58}$$

where the terms $x_{\alpha\beta}(t)$ are "adapted" in the formal sense that they do not depend on the noises $b_k(s)$ for $s>t$. As we are talking about the vacuum representation for the time being, we can bootstrap from the vacuum $|\Omega\rangle$ to construct the Fock space as the completion of the span of all vectors of the type $\int f_{k(1)}(t_1)\, b_{k(1)}(t_1)^* \cdots f_{k(n)}(t_n)\, b_{k(n)}(t_m)^* |\Omega\rangle$, and moreover we can build up the domain of exponential vectors. We quickly see that (58), with Wick ordered right hand side, corresponds to the QSDE

$$dX(t) = x_{jk}(t)\, d\Lambda_{lk}(t) + x_{j0}(t)\, dB_j(t)^* + x_{0k}(t)\, dB_k(t) + x_{00}(t)\, dt. \tag{59}$$

Our issue however is how do we put to Wick order a given expression, for instance, the right hand side of (57).

Proposition 5 *Proposition For the process $X(t)$ described by (58), we have*

$$\begin{aligned} b_k(t)\, X(t) &= X(t)\, b_k(t) + \frac{1}{2} x_{kl}(t)\, b_l(t) + \frac{1}{2} x_{k0}(t), \\ X(t)\, b_k(t)^* &= b_k(t)^* X(t) + \frac{1}{2} b_j(t)^* x_{j0}(t) + \frac{1}{2} x_{0k}(t). \end{aligned} \tag{60}$$

We may justify this as follows:

$$\begin{aligned} [b_k(t), X(t)] &= \int_0^t \left[b_k(t), \dot{X}(s) \right] ds = \int_0^t \delta(t-s) \{ x_{kl}(s)\, b_l(s) + x_{k0}(s) \} \\ &= \frac{1}{2} x_{kl}(t)\, b_l(t) + \frac{1}{2} x_{k0}(t) \end{aligned} \tag{61}$$

with the factor of $\frac{1}{2}$ coming from the half-contribution of the δ-function. Evidently what the equations in (60) correspond to is our definition of a Stratonovich differential - at least for the Fock vacuum representation. While we can make a connection between (58) and the rigorously defined Hudson-Parthasarathy processes, it should be appreciated at the very least that (60) is the correct mnemonic for doing the Wick ordering - an attempt to convert into a Dyson-type series expansion and Wick ordering under the iterated integral signs to get a Maassen-Meyer kernel expansion shows this. At work here is an old principle that "Itôs formula is the chain rule with Wick ordering" [16]. Let us now examine (57) and put it to Wick ordered form. By a similar argument, we have

$$[b_k(t), U(t)] = \int_0^t [b_k(t), \Upsilon(s)]\, U(s)\, ds \equiv \frac{1}{2} L_k U(t), \tag{62}$$

or $b_k(t)\, U(t) = U(t)\, b_k(t) + \frac{1}{2} L_k U(t)$. By means of this we may place (57) into the Wick-ordered form

$$U(t) = L_k b_k^*(t)\, U(t) - L_k^* U(t)\, b_k(t) - (\frac{1}{2} L_k^* L_k + iH) U(t), \tag{63}$$

and picking up the correct vacuum damping (9), K, as a result.

Setting $X_t = U(t)(X \otimes I)U(t)$, the same Wick ordering rule can be applied to the Heisenberg equations to obtain

$$\dot{X}_t = \left\{ b_k^*(t) + \frac{1}{2} L_{k,t}^* \right\} [X, L_k]_t + [L_k^*, X]_t \left\{ b_k(t) + \frac{1}{2} L_{k,t} \right\} + \frac{1}{i} U(t) [X, H] U(t) . \tag{64}$$

Here we use the notation $L_{k,t} = U(t)(L_k \otimes I)U(t)$, etc.

We also remark that we may define the corresponding *output fields* by

$$b_k^{\text{out}}(t) \triangleq U_T^* \, b(t) \, U_T, \tag{65}$$

where $T > t$. One may show that the input-output relations are

$$b_k^{\text{out}}(t) \equiv b_k(t) + L_{k,t}. \tag{66}$$

If, on the other hand, we want the state of the noise to be a mean-zero Gaussian with correlations, say

$$\left\langle b_j(t)^* \, b_k(s) \right\rangle = n_{jk} \, \delta(t-s) , \quad \left\langle b_j(t) \, b_k(s) \right\rangle = m_{jk} \, \delta(t-s) , \tag{67}$$

then we represent the noise as

$$b_k(t) = U_{jk} a_{+,k}(t) + V_{jk} a_{-,k}(t)^* \tag{68}$$

employing a suitable Bogoliubov transformation. Here we now have double the number of quantum white noises $a_{+,k}$ and $a_{-,k}$ but these are represented as Fock processes.

If we now substitute (68) into (57) we see explicitly that the $a_{\pm,k}$ are out Wick order, but this can be rectified by the same sort of manipulation as above. Once the $a_{\pm,k}(t)$ are Wick ordered, we have a equation which we can interpret as the Itō non-Fock QSDE, and this leads to the correct expressions $K^{(N,M)}$ and $\mathcal{L}^{(N,M)}$ in the unitary and flow equations respectively.

Given a Gaussian state $\langle \cdot \rangle$ on the noise, we may introduce a conditional expectation according to $\mathbb{E}[\cdot|\text{sys}] : A \otimes B \mapsto \langle B \rangle \, A$. For instance, $\mathbb{E}[U(t)|\text{sys}]$ then defines a contraction on the system Hilbert space and we have

$$\mathbb{E}[U(t)|\text{sys}] = I_{\text{sys}} + \sum_{n \geq 1} (-i)^n \int_{\Delta_n(t)} \mathbb{E}[\Upsilon_{s_n} \cdots \Upsilon_{s_1}|\text{sys}] . \tag{69}$$

Now the expression $\mathbb{E}[\Upsilon_{s_n} \cdots \Upsilon_{s_1}|\text{sys}]$ will be a sum of products of the operators $L, -L^*$ and H times a n-point function in the fields. Similarly, we obtain a reduced Heisenberg equation. To compute these averages we need to be able to calculate n-point functions of chronologically ordered Gaussian fields - this is the realm of Wick's Theorem, so what we have presented may be interpreted as a Gaussian Wick's Theorem [26]. We of course recover the partial traces appearing in the previous section.

6 Approximate Signal Generator for Thermal States

In this section we show how to go from a general SLH model driven by the output of a Degenerate Parametric Amplifier (DPA) to the limit where the same SLH model is driven by a thermal white noise. We start with the single channel for simplicity.

6.1 The Thermal White Noise as Idealization of the Output of a Degenerate Parametric Amplifier

We now show that in the strong coupling limit the output of a degenerate parametric amplifier approximates a thermal white noise. the model consists of a system of two cavities modes c_+ and c_- coupled to input processes $A_+(t)$ and $A_-(t)$ respectively. Both inputs are taken to be in the vacuum state and the Schrödinger equation is

$$\dot{U}_t = \sum_{i=+,-} L_i U(t)\, dA_i(t)^* - \sum_{i=+,-} L_i^* U(t)\, dA_i(t) - iH_{\text{amp}} U_t, \tag{70}$$

with initial condition $U_0 = I$ and

$$L_+ = \sqrt{2\kappa k} c_+, \quad L_- = \sqrt{2\kappa k} c_- \text{ and } H_{\text{amp}} = \frac{\varepsilon k}{i}\left(c_+ c_- - c_+ c_-\right). \tag{71}$$

Here $\varepsilon > \kappa$ and $k > 0$ is a scaling parameter which we eventually model to be large. It is more convenient to work with the white noises $a_\pm(t)$.

The model is linear and we obtain the input-output relations in the Laplace domain to be [10]

$$\begin{bmatrix} b[s] \\ \tilde{b}[s] \end{bmatrix} = \Xi_-^{(k)}(s) \begin{bmatrix} a_+[s] \\ a_-[s] \end{bmatrix} + \Xi_+^{(k)}(s) \begin{bmatrix} a_+[s] \\ a_-[s] \end{bmatrix} \tag{72}$$

where $\Xi_-^{(k)}(s) = \begin{bmatrix} u(s/k) & 0 \\ 0 & u(s/k) \end{bmatrix}$, $\Xi_+^{(k)}(s) = \begin{bmatrix} 0 & v(s/k) \\ v(s/k) & 0 \end{bmatrix}$ with the functions $u(s) = \frac{s^2-\kappa^2-\varepsilon^2}{s^2+2s\kappa+\kappa^2-\varepsilon^2}$, $v(s) = \frac{2\kappa\varepsilon}{s^2+2\kappa+\kappa^2-\varepsilon^2}$.

In the limit $k \to \infty$ we find the static (s-independent) coefficients

$$\lim_{k\to\infty} \Xi_-^{(k)}(s) = \frac{\varepsilon^2+\kappa^2}{\varepsilon^2-\kappa^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lim_{k\to\infty} \Xi_+^{(k)}(s) = \frac{2\varepsilon\kappa}{\varepsilon^2-\kappa^2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{73}$$

and returning to the time domain, the limit output fields are just a Bogoliubov transform of the inputs

$$b(t) = \sqrt{n+1}\, a_+(t) + \sqrt{n}\, a_-(t), \quad \tilde{b}(t) = \sqrt{n}\, a_+(t) + \sqrt{n+1}\, a_-(t), \tag{74}$$

Here the parameter n corresponds is $n = \left(\frac{2\varepsilon\kappa}{\varepsilon^2-\kappa^2}\right)^2$.

It is instructive to look closely at the finite k equations. We have the Heisenberg equations

$$\begin{aligned}\dot{c}_+ (t) &= -k\kappa c_+ (t) + k\varepsilon c_- (t) - \sqrt{2\kappa k} a_+ (t) , \\ \dot{c}_- (t) &= -k\kappa c_- (t) + k\varepsilon c_+ (t) - \sqrt{2\kappa k} a_- (t) ,\end{aligned} \tag{75}$$

and for k large we may ignore the $\dot{c}_+ (t)$ and $\dot{c}_- (t)$ terms leaving a pair of simultaneous equations which we may solve to get

$$\begin{aligned}\sqrt{k} c_+ (t) &\simeq \frac{\sqrt{2\kappa}}{\varepsilon^2 - \kappa^2} \left[\kappa a_+ (t) + \varepsilon a_- (t)^* \right] , \\ \sqrt{k} c_- (t) &\simeq \frac{\sqrt{2\kappa}}{\varepsilon^2 - \kappa^2} \left[\kappa a_- (t) + \varepsilon a_+ (t)^* \right] .\end{aligned} \tag{76}$$

The output is then

$$\begin{aligned}b (t) &= a_+ (t) + \sqrt{2\kappa k} c_+ (t) \simeq a_+ (t) + \frac{2\kappa}{\varepsilon^2 - \kappa^2} \left[\kappa a_+ (t) + \varepsilon a_- (t)^* \right] \\ &\equiv \sqrt{n+1} a_+ (t) + \sqrt{n} a_- (t) ,\end{aligned} \tag{77}$$

and likewise

$$\begin{aligned}\tilde{b} (t) &= a_- (t) + \sqrt{2\kappa k} c_- (t) \simeq a_- (t) + \frac{2\kappa}{\varepsilon^2 - \kappa^2} \left[\kappa a_- (t) + \varepsilon a_+ (t)^* \right] \\ &\equiv \sqrt{n} a_+ (t) + \sqrt{n+1} a_- (t) .\end{aligned} \tag{78}$$

It is relatively straightforward to find a multi-dimensional version of this for a general Bogoliubov transformation

$$\begin{bmatrix} b (t) \\ \tilde{b} (t) \end{bmatrix} = U \begin{bmatrix} a_+ (t) \\ a_- (t) \end{bmatrix} + V \begin{bmatrix} a_+ (t) \\ a_- (t) \end{bmatrix} . \tag{79}$$

6.2 Cascade Approximation

The DPA which is described by

$$\mathbf{G}_{DPA} \sim \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \begin{bmatrix} \sqrt{2\kappa k} c_+ \\ \sqrt{2\kappa k} c_- \end{bmatrix} , H_{\text{amp}} \right) \tag{80}$$

driven by the (vacuum) input pair $\begin{bmatrix} a_+ (t) \\ a_- (t) \end{bmatrix}$. It is then put in series with

$$\mathbf{G} \sim (S, L, H) \boxplus (1, 0, 0) = \left(\begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} , \begin{bmatrix} L \\ 0 \end{bmatrix} , H \right) \tag{81}$$

which means that the output $a_+ (t)$ is fed in as input to the system $\mathbf{G} \sim (S, L, H)$ and $a_- (t)$ is left to go away unhindered, $\mathbf{G}_{\text{trivial}} \sim (1, 0, 0)$. According to the series product rule, we get DPA and system in series is described by,

$$\mathbf{G} \triangleleft \mathbf{G}_{DPA} \sim \left(\begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} , \begin{bmatrix} L + S\sqrt{2\kappa k} c_+ \\ \sqrt{2\kappa k} c_- \end{bmatrix} , H + H_{\text{amp}} + \frac{\sqrt{\kappa k}}{\sqrt{2} i} \left(L^* S c_+ - c_+^* S^* L \right) \right) . \tag{82}$$

From this we obtain the Heisenberg equations

$$\begin{aligned}\dot{X}_t &= a_+(t)^* (S^* X S - X)_t a_+(t) + a_+(t)^* S_t^* [X, L]_t + [L^*, X]_t S_t a_+(t) \\ &\quad + \frac{1}{2} [L^*, X]_t \left(L + S\sqrt{2\kappa k} c_+ \right)_t + \frac{1}{2} \left(L + S\sqrt{2\kappa k} c_+ \right)_t^* [X, L]_t \\ &\quad - i \left[X, H + \frac{\sqrt{2\kappa k}}{2i} \left(L^* S c_+ - c_+^* S^* L \right) \right]_t . \end{aligned} \tag{83}$$

We now make the approximation $\sqrt{k} c_+(t) \simeq \frac{\sqrt{2\kappa}}{\varepsilon^2 - \kappa^2} \left[\kappa a_+(t) + \varepsilon a_-(t)^* \right]$ which leads to

$$\begin{aligned}\dot{X}_t &\simeq a_+(t)^* (S^* X S - X)_t a_+(t) + a_+(t)^* S_t^* [X, L]_t + [L^*, X]_t S_t a_+(t) + \mathcal{L}(X)_t \\ &\quad + \left\{ [L^*, X]_t S_t + \frac{1}{2} L_t^* [S, X]_t \right\} \left[\left(\sqrt{n+1} - 1 \right) a_+(t) + \sqrt{n} a_-(t)^* \right] \\ &\quad + \left[\left(\sqrt{n+1} - 1 \right) a_+(t)^* + \sqrt{n} a_-(t) \right] \left\{ S_t^* [X, L]_t + \frac{1}{2} [X, S^*]_t L_t \right\} . \end{aligned} \tag{84}$$

Here we have $n = \left(\frac{2\varepsilon\kappa}{\varepsilon^2 - \kappa^2} \right)^2$, as before.

We now make a key assumption: **the scattering term S corresponds to a static element**. In this case $S \equiv e^{i\theta}$ for some real θ. The limit Heisenberg equation therefore simplifies to

$$\begin{aligned}\dot{X}_t &= a_+(t)^* S^* [X, L]_t + [L^*, X]_t S a_+(t) + \mathcal{L}(X)_t \\ &\quad + [L^*, X]_t S \left[\left(\sqrt{n+1} - 1 \right) a_+(t) + \sqrt{n} a_-(t)^* \right] \\ &\quad + \left[\left(\sqrt{n+1} - 1 \right) a_+(t)^* + \sqrt{n} a_-(t) \right] S^* [X, L]_t \\ &= \sqrt{n+1} a_+(t)^* S^* [X, L]_t + \sqrt{n+1} [L^*, X]_t S a_+(t) \\ &\quad + \sqrt{n} [L^*, X]_t S a_-(t)^* + \sqrt{n} a_-(t) S^* [X, L]_t + \mathcal{L}(X)_t . \end{aligned} \tag{85}$$

We are not quite finished as the operators $a_-(t)$ and $a_-(t)$ are out of Wick order. However, this is easily remedied. For instance, we easily deduce that

$$\begin{aligned}\left[Y_t, a_-(t)^* \right] &= \int_0^t \left[\dot{Y}_s, a_-(t)^* \right] ds \\ &= \int_0^t \left[\sqrt{n} a_-(s) S^* [Y, L]_s , a_-(t)^* \right] ds \\ &= \frac{1}{2} \sqrt{n} S^* [Y, L]_t \end{aligned} \tag{86}$$

so that we arrive at

$$[L^*, X]_t S a_-(t)^* = a_-(t)^* [L^*, X]_t S + \frac{1}{2} \sqrt{n} \left[[L^*, X], L \right]_t . \tag{87}$$

Similarly $\left[a_{-}(t), Y_t\right] = \frac{1}{2}\sqrt{n}\left[L^*, Y\right]_t$ and therefore we get the Wick re-ordering

$$a_{-}(t) S^* [X, L]_t = S^* [X, L]_t a_{-}(t) + \frac{1}{2}\sqrt{n} [L^*, [X, L]]_t . \tag{88}$$

This leads to the form of the quantum white noise equation with both a_+ and a_- Wick ordered as

$$\begin{aligned}
\dot{X}_t &= \sqrt{n+1} a_+(t)^* S^* [X, L]_t + \sqrt{n+1} [L^*, X]_t S a_+(t) \\
&\quad + \sqrt{n} a_-(t)^* [L^*, X]_t S + \sqrt{n} S^* [X, L]_t a_-(t) \\
&\quad + \mathcal{L}(X)_t + \frac{1}{2} n [[L^*, X], L]_t + \frac{1}{2} n [L^*, [X, L]]_t .
\end{aligned} \tag{89}$$

At this stage we recognize (89) as the equivalent form of the Heisenberg quantum stochastic differential equation for thermal noise.

We also remark that the output process determined by systems in series is $B^{\text{out}}(t) = U_t^* A_+(t) U_t$, and from the quantum stochastic calculus we have

$$dB^{\text{out}}(t) = dA_+(t) + \left(L + S\sqrt{2\kappa k} c_+\right)_t dt. \tag{90}$$

Using (76) we approximate this as

$$dB^{\text{out}}(t) \simeq dA_+(t) + L_t dt + S\frac{2\kappa}{\varepsilon^2 - \kappa^2}\left[\kappa dA_+(t) + \varepsilon dA_-(t)^*\right] \equiv S dB^{\text{in}}(t) + L_t dt, \tag{91}$$

that is, the thermal input $B^{\text{in}}(t) = \sqrt{n+1} A_+(t) + \sqrt{n} A_-(t)^*$ produces the output $B^{\text{out}}(t)$ according to the usual rules one would expect of a quantum Markov component with the parameters $\mathbf{G} \sim (S, L, H)$.

Therefore the description of a component with the parameters $\mathbf{G} \sim (S, L, H)$, at least in the case where S is a static beam-splitter matrix, with Gaussian input processes may be considered as the same component cascaded with a degenerate parametric amplifier with vacuum inputs in the singular coupling limit of the DPA.

7 The General Series Product

7.1 Without Scattering

Let us now consider the situation where a Gaussian input $B_{\text{in}} = B_{\text{in}}^{(\mathscr{A})}$ is driving a system with SLH parameters $(I, L_{\mathscr{A}}, H_{\mathscr{A}})$ and that its output $B_{\text{out}}^{(\mathscr{A})}$ acts as input $B_{\text{in}}^{(\mathscr{B})}$ to a second system $(I, L_{\mathscr{B}}, H_{\mathscr{B}})$. (We do not assume that any of the various SLH operators commute!)

(Components in Series: The no scattering case) The Heisenberg QSDE for the systems $(I, L_{\mathscr{A}}, H_{\mathscr{A}})$ and $(I, L_{\mathscr{B}}, H_{\mathscr{B}})$ given by

$$dj_t(X) = \sum_{\mathscr{S}=\mathscr{A},\mathscr{B}} \left\{ dB_{\text{in}}^{(\mathscr{S})*} \circ j_t([X, L_{\mathscr{S}}]) + j_t([L_{\mathscr{S}}^*, X]) \circ dB_{\text{in}}^{(\mathscr{S})} + j_t(\mathcal{L}_{\mathscr{S}} X) \circ dt \right\}, \tag{92}$$

where

$$\mathcal{L}_{\mathscr{S}} X = \frac{1}{2} L_{\mathscr{S}}^* [X, L_{\mathscr{S}}] + \frac{1}{2} [L_{\mathscr{S}}^*, X] L_{\mathscr{S}} - i [X, H_{\mathscr{S}}]. \tag{93}$$

and we have the constraints $B_{\text{in}}^{(\mathscr{A})} = B_{\text{in}}$ and $dB_{\text{in}}^{(\mathscr{B})} = dB_{\text{in}}^{(\mathscr{A})} + j_t(L_{\mathscr{A}}) \, dt$, consistent with B_{in} driving system $\mathscr{A}$ which in turn drives $\mathscr{B}$, corresponds to the dynamics given by the intrinsic series product (22).

Proof. We have to show consistency of the quantum stochastic Heisenberg evolution $j_t(\cdot)$. To this end we take the open loop equations and impose the constraint $dB_{\text{in}}^{(\mathscr{B})} = dB_{\text{in}}^{(\mathscr{A})} + j_t(L_{\mathscr{A}}) \, dt$ giving

$$\begin{aligned} dj_t(X) &= dB_{\text{in}}^* \circ j_t([X, L_{\mathscr{A}}]) + j_t([L_{\mathscr{A}}^*, X]) \circ dB_{\text{in}} \\ &+ (dB_{\text{in}} + j_t(L_{\mathscr{A}}))^* \circ j_t([X, L_{\mathscr{B}}]) \\ &+ j_t([L_{\mathscr{B}}^*, X]) \circ (dB_{\text{in}} + j_t(L_{\mathscr{A}}) dt) \\ &+ j_t(\mathcal{L}_{\mathscr{A}} X) \circ dt + j_t(\mathcal{L}_{\mathscr{B}} X) \circ dt, \end{aligned} \tag{94}$$

which we may rearrange as

$$\begin{aligned} dj_t(X) &= dB_{\text{in}}^* \circ j_t([X, L_{\mathscr{A}} + L_{\mathscr{B}}]) + j_t([L_{\mathscr{A}}^* + L_{\mathscr{B}}^*, X]) \circ dB_{\text{in}} \\ &\quad + j_t \left(\mathcal{L}_{\mathscr{A}} X + \mathcal{L}_{\mathscr{B}} X + L_{\mathscr{A}}^* [X, L_{\mathscr{B}}] + [L_{\mathscr{B}}^*, X] L_{\mathscr{A}} \right) \circ dt. \end{aligned} \tag{95}$$

However, the dt term can be recast using the identity

$$\begin{aligned} & \mathcal{L}_{\mathscr{A}} X + \mathcal{L}_{\mathscr{B}} X + L_{\mathscr{A}}^* [X, L_{\mathscr{B}}] + [L_{\mathscr{B}}^*, X] L_{\mathscr{A}} \\ = & \frac{1}{2} (L_{\mathscr{A}} + L_{\mathscr{B}})^* [X, L_{\mathscr{A}} + L_{\mathscr{B}}] + \frac{1}{2} [L_{\mathscr{A}}^* + L_{\mathscr{B}}^*, X] (L_{\mathscr{A}} + L_{\mathscr{B}}) \\ & -i \left[X, H_{\mathscr{A}} + H_{\mathscr{B}} + \frac{1}{2i} (L_{\mathscr{B}}^* L_{\mathscr{A}} - L_{\mathscr{A}}^* L_{\mathscr{B}}) \right]. \end{aligned} \tag{96}$$

The resulting Heisenberg dynamics is therefore the same as for the model (I, L, H) with $L = L_{\mathscr{A}} + L_{\mathscr{B}}$, and $H = H_{\mathscr{A}} + H_{\mathscr{B}} + \text{Im}\{L_{\mathscr{B}}^* L_{\mathscr{A}}\}$. This is, of course, the form predicted by the series product in the Fock case (22). ■

7.2 Including Scattering

As mentioned above, it is not possible to construct a well defined scattering processes Λ_{jk} in the non-Fock theory. Nevertheless, the effects of static beam-splitter scattering S may be included in a straightforward manner without directly considering unitary QSDE models involving the scattering processes. A clue on how to proceed is given by our earlier observation that if the scattering matrix S entries commute with systems operators - physically, a static beam-splitter - the scattering processes disappears.

In the Fock representation, we could always take the input field $A_{\rm in}$ and apply a unitary rotation $A = SA_{\rm in}$ before passing it though as drive for component. As we have seen, this will require a compensating rotation of the coupling operators, but no change to the Lindbladian. There is also a rotation of the output, however, anticipating this we make the following definition.

Definition 6 *Definition Let* $\mathbf{G}$ *and* $\tilde{\mathbf{G}}$ *be SLH model parameters which, for given input noise* $A_{\rm in} = \tilde{A}_{\rm in}$ *lead to output noises* $A_{\rm out}$ *and* $\tilde{A}_{\rm out}$ *respectively. We say that the models' input-output relations are* ***related by a static beam-splitter matrix*** S *if we have*

$$A_{\rm out} = S\,\tilde{A}_{\rm out}. \tag{97}$$

The following result shows that for the Fock representation, if the scattering is just a static beam-splitter, then we can produce a related model which avoids the use of the scattering processes.

Theorem 7 *Theorem Let* S *be a static beam-splitter matrix and set* $\mathbf{G} \sim (S, L, H)$ *and* $\tilde{\mathbf{G}} \sim (I, S^* L, H)$*. Then the model parameters* $\mathbf{G}$ *and* $\tilde{\mathbf{G}}$ *generate the same Heisenberg dynamics. Moreover, their input-output relations are related by the static beam-splitter matrix* S*.*

Proof. The Heisenberg dynamics generated by $\mathbf{G}$ is (the scattering terms vanish for a static beam-splitter)

$$dj_t^{\mathbf{G}}(X) = \sum_j j_t(\mathcal{L}^{\mathbf{G}}_{j0} X)\, dA_j^* + \sum_k j_t(\mathcal{L}^{\mathbf{G}}_{0k} X)\, dA_k + j_t(\mathcal{L}^{\mathbf{G}} X) dt \tag{98}$$

where

$$\mathcal{L}^{\mathbf{G}}_{j0} X = S^*_{lj}\, [X, L_l]\,, \quad \mathcal{L}^{\mathbf{G}}_{0k} X = [L_l^*, X]\, S_{lk} \tag{99}$$

and the Lindblad generator is $\mathcal{L}^{\mathbf{G}} X = \frac{1}{2} L_k^* [X, L_k] + \frac{1}{2} [L_k^*, X]\, L_k - i\,[X, H]$. The Heisenberg dynamics for $\tilde{\mathbf{G}}$ similarly has no scattering terms in its QSDE, and we see that

$$\mathcal{L}^{\mathbf{G}}_{j0} X = [X, S^*_{lj} L_l] \equiv \mathcal{L}^{\tilde{\mathbf{G}}}_{j0} X, \quad \mathcal{L}^{\mathbf{G}}_{0k} X = [L_l^* S_{lk}, X] \equiv \mathcal{L}^{\tilde{\mathbf{G}}}_{0k} X. \tag{100}$$

From the unitarity and scalar nature of S we have that

$$\begin{aligned} \mathcal{L}^{\tilde{\mathbf{G}}} X &= \frac{1}{2} L_k^* S_{kl} \left[X, S^*_{jl} L_j\right] + \frac{1}{2} \left[L_k^* S_{kl}, X\right] S^*_{jl} L_j - i\,[X, H] \\ &= \frac{1}{2} L_k^* [X, L_k] + \frac{1}{2} [L_k^*, X]\, L_k - i\,[X, H] \\ &\equiv \mathcal{L}^{\mathbf{G}} X. \end{aligned} \tag{101}$$

Therefore the QSDEs corresponding to the Heisenberg dynamics for $\mathbf{G}$ and $\tilde{\mathbf{G}}$ are identical.

The input-output relations for $\mathbf{G}$ are

$$dA_{\text{out},j}\left(t\right)=S_{jk}\,dA_{\text{in},k}+j_{t}\left(L_{j}\right)\,dt \tag{102}$$

while for $\tilde{\mathbf{G}}$ we have

$$dB_{\text{out},j}\left(t\right)=dB_{\text{in},j}+S_{jk}\,j_{t}\left(L_{k}\right)\,dt. \tag{103}$$

If we require the inputs to be the same ($A_{\text{in}}=B_{\text{in}}$) then we have $A_{\text{out}}=S\,B_{\text{out}}$. ■

Our strategy for introducing static beam-splitter scattering into the situation where we have non-Fock noise input fields is to say that the initial input A_{in} be replaced by the rotated input SA_{in}, and exploit the fact that the Heisenberg dynamics no longer involves the scattering processes Λ_{jk} explicitly.

Lemma 8 *Lemma **(The Universal Heisenberg QSDE Description)** The Heisenberg dynamics for a general (S,L,H) model with a static beam-splitter matrix S are given by the QSDE*

$$dj_{t}(X) \quad = \quad dA_{\text{in}}^{\ast}\circ S^{\ast}j_{t}([X,L])+j_{t}([L^{\ast},X])S\circ dA_{\text{in}}+j_{t}(\mathcal{L}X)\circ dt \tag{104}$$

for all mean-zero Gaussian input fields A_{in}.

This is of course just the equation (89) written in the Wick-Stratonovich form so as to be representation free!

Now let us try and repeat or analysis from Section 7.1. Let us now consider the situation where a Gaussian input $A_{\text{in}}=A_{\text{in}}^{(1)}$ is driving a system with SLH parameters $(S_{\mathscr{A}},L_{\mathscr{A}},H_{\mathscr{A}})$ and that its output $A_{\text{out}}^{(1)}$ acts as input for a second system $(S_{\mathscr{B}},L_{\mathscr{B}},H_{\mathscr{B}})$.

Lemma 9 *Lemma **(Components in series: With a static beam-splitter scattering)** The Heisenberg QSDE for a pair of systems $(S_{\mathscr{A}},L_{\mathscr{A}},H_{\mathscr{A}})$ and $(S_{\mathscr{B}},L_{\mathscr{B}},H_{\mathscr{B}})$ in series is*

$$dj_{t}(X)=\sum_{\mathscr{S}=\mathscr{A},\mathscr{B}}\left\{dA_{\text{in}}^{(\mathscr{S})\ast}\circ j_{t}([X,L_{\mathscr{S}}])+j_{t}([L_{\mathscr{S}}^{\ast},X])\circ dA_{\text{in}}^{(\mathscr{S})}+j_{t}(\mathcal{L}_{\mathscr{S}}X)\circ dt\right\} \tag{105}$$

where $A_{\text{in}}^{(\mathscr{A})}=S_{\mathscr{A}}A_{\text{in}}$ and $A_{\text{in}}^{(\mathscr{B})}=S_{\mathscr{B}}A_{\text{out}}^{(\mathscr{A})}$ where $dA_{\text{out}}^{(\mathscr{A})}=S_{\mathscr{A}}dA_{\text{in}}^{(\mathscr{A})}+j_{t}\left(L_{\mathscr{A}}\right)dt$, and the Lindbladians $\mathcal{L}_{\mathscr{S}}$ are as before.

Proof. Substituting the processes into the QSDEs yields

$$\begin{aligned} dj_t(X) &= (S_{\mathscr{A}} dA_{\text{in}})^* \circ j_t([X, L_{\mathscr{A}}]) + j_t([L_{\mathscr{A}}^*, X]) \circ S_{\mathscr{A}} dA_{\text{in}} \\ &\quad + (S_{\mathscr{B}} S_{\mathscr{A}} dA_{\text{in}} + S_{\mathscr{B}} L_{\mathscr{A}} dt)^* \circ j_t([X, L_{\mathscr{B}}]) \\ &\quad + j_t([L_{\mathscr{B}}^*, X]) \circ (S_{\mathscr{B}} S_{\mathscr{A}} dA_{\text{in}} + S_{\mathscr{B}} L_{\mathscr{A}} dt) \\ &\quad + j_t(\mathcal{L}_{\mathscr{A}} X) \circ dt + j_t(\mathcal{L}_{\mathscr{B}} X) \circ dt, \\ &= (dA_{\text{in}})^* \circ j_t([X, S_{\mathscr{A}}^* L_{\mathscr{A}} + S_{\mathscr{A}}^* S_{\mathscr{B}}^* L_{\mathscr{B}}]) + j_t([L_{\mathscr{A}}^* S_{\mathscr{A}} + L_{\mathscr{B}}^* S_{\mathscr{B}} S_{\mathscr{A}}, X]) \circ dA_{\text{in}} \\ &\quad + j_t(\mathcal{L}_{\mathscr{A}} X + \mathcal{L}_{\mathscr{B}} X + L_{\mathscr{A}}^* S_{\mathscr{B}}^* [X, L_{\mathscr{B}}] + [L_{\mathscr{B}}^*, X] S_{\mathscr{B}} L_{\mathscr{A}}) \circ dt. \end{aligned} \tag{106}$$

A similar calculation to before shows that

$$\begin{aligned} & \mathcal{L}_{\mathscr{A}} X + \mathcal{L}_{\mathscr{B}} X + L_{\mathscr{A}}^* S_{\mathscr{B}}^* [X, L_{\mathscr{B}}] + [L_{\mathscr{B}}^*, X] S_{\mathscr{B}} L_{\mathscr{A}} \\ = \; & \frac{1}{2} (S_{\mathscr{B}} L_{\mathscr{A}} + L_{\mathscr{B}})^* [X, S_{\mathscr{B}} L_{\mathscr{A}} + L_{\mathscr{B}}] + \frac{1}{2} [L_{\mathscr{A}}^* S_{\mathscr{B}}^* + L_{\mathscr{B}}^*, X] (S_{\mathscr{B}} L_{\mathscr{A}} + L_{\mathscr{B}}) \\ & - [iX, H_{\mathscr{A}} + H_{\mathscr{B}} + \frac{1}{2i} (L_{\mathscr{B}}^* S_{\mathscr{B}} L_{\mathscr{A}} - L_{\mathscr{A}}^* S_{\mathscr{B}}^* L_{\mathscr{B}})]. \end{aligned} \tag{107}$$

The resulting Heisenberg dynamics is therefore same as for the model $\tilde{\mathbf{G}} \sim (I, \tilde{L}, H)$ with coupling operators $\tilde{L} = S_{\mathscr{A}}^* L_{\mathscr{A}} + S_{\mathscr{A}}^* S_{\mathscr{B}}^* L_{\mathscr{B}} \equiv S_{\mathscr{A}}^* S_{\mathscr{B}}^* (S_{\mathscr{B}} L_{\mathscr{A}} + L_{\mathscr{B}})$, and Hamiltonian $H = H_{\mathscr{A}} + H_{\mathscr{B}} + \text{Im}\{L_{\mathscr{B}}^* S_{\mathscr{B}} L_{\mathscr{A}}\}$.

The output is then B_{out} where

$$dB_{\text{out}}(t) = dA_{\text{in}}(t) + j_t (S_{\mathscr{A}}^* L_{\mathscr{A}} + S_{\mathscr{A}}^* S_{\mathscr{B}}^* L_{\mathscr{B}}) \, dt. \tag{108}$$

The correct output for this should however be $A_{\text{out}} = S_{\mathscr{B}} S_{\mathscr{A}} B_{\text{out}}$ so that

$$dA_{\text{out}}(t) = S_{\mathscr{B}} S_{\mathscr{A}} dA_{\text{in}}(t) + j_t (S_{\mathscr{B}} L_{\mathscr{A}} + L_{\mathscr{B}}) \, dt \tag{109}$$

and we have the desired matrix $S_{\mathscr{B}} S_{\mathscr{A}}$ multiply the inputs corresponding to scattering first by matrix $S_{\mathscr{A}}$ and then by $S_{\mathscr{B}}$. The model $\mathbf{G}$ obtained from postulate Ia is then the one related to $\tilde{\mathbf{G}}$ by the static beam-splitter matrix $S_{\mathscr{B}} S_{\mathscr{A}}$, that is (from Theorem 7 with $S = S_{\mathscr{B}} S_{\mathscr{A}}$ and $\tilde{L} = S^* L$

$$\begin{aligned} \mathbf{G} &\sim (S, L, H) = \left(S_{\mathscr{B}} S_{\mathscr{A}}, S_{\mathscr{B}} S_{\mathscr{A}} \tilde{L}, H \right) \\ &= (S_{\mathscr{B}} S_{\mathscr{A}}, S_{\mathscr{B}} L_{\mathscr{A}} + L_{\mathscr{B}}, H_{\mathscr{A}} + H_{\mathscr{B}} + \text{Im}\{L_{\mathscr{B}}^* S_{\mathscr{B}} L_{\mathscr{A}}\}), \end{aligned}$$

and again we have the same form as the series product in the Fock case (22). ■

8 Conclusions

We have shown that there is a consistent theory for quantum input-output models in series when the driving input processes are in general Gaussian states with a flat power spectrum. This emerges fairly explicitly at the level of the singular input processes $b_k(t)$ themselves, but to have a working theory we need to make the connection to the Hudson-Parthasarathy quantum stochastic calculus. This involves quantum stochastic differential equations on the Fock spaces used to represent the noise (which are a mathematical convenience and not physical objects) with the result that the associated dynamical equations appear to depend on the choice of Gaussian state of the noise. In reality this is a mathematical artifact and we show that even here there is a way of expressing the quantum stochastic differential equations (the Wick-Stratonovich form introduced in this paper) which removes these terms. In effect, it is the Wick-Stratonovich form that translates in the physically relevant dynamical equations written in terms of the quantum input processes $b_k(t)$.

The connection rules are then shown to be genuinely independent of the choice of state. We were also able to include the effects of a static beam-splitter component. At first sight this would seem problematic as the scattering terms $\Lambda_{jk}(t)$ are not well-defined for non-vacuum states, however, it is possible to ignore them from the model: in fact we need to work at the level of the Heisenberg flow and the input-output relations, neither of which involve the scattering terms. The result is that we may account for static scattering and we find that the series product of [9] again gives the correct rule. In this way we extend the series product to deal with quantum feedback networks driven by general Gaussian input processes.

We have restricted our analysis to Bose systems, however, there is an Araki-Woods type double Fock space representation for Fermi fields with quasi-free states as well, and is applicable to Fermi stochastic processes [27], [28]. The network rules for Fermi stochastic processes can be similarly derived and one would naturally expect these to again be state-independent.

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